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"Free" Evolution of Multi-particle Excitations in the Glauber Dynamics at High Temperature

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The paper is devoted to the spectral properties of the Glauber dynamics for the Ising model at high temperature $1/\beta$. It is proven that for sufficiently small $|\beta|$ there is an invariant subspace, in which the dynamics can be described as "a free multi-particle evolution", provided that the one-particle subspace is singled out. The proof is based upon the Haag–Ruelle scattering theory.

KEY WORDS: Generator of the Glauber dynamics; k-particle subspace; second quantization; wave operator.

1. INTRODUCTION AND MAIN RESULTS

It is generally accepted that the operators describing the evolution of infinite-component translation-invariant systems with local interactions possess a so-called "corpuscular" spectral picture. This means that there is a number of invariant subspaces, cyclic with respect to the translation group, which are called "one-particle subspaces"; all the other ("k-particle") branches of the spectrum are wholly determined by finite sets of these "particles" (details are explained below). For the time being this corpuscular picture has been completely or even partially established only for several models (see ref. 1–6). The present paper is also devoted to this problem, though our method of the construction of the multi-particle subspaces is different from that of the aforementioned papers. Namely, starting from the already constructed one-particle subspace, we proceed to the k-particle subspaces using a well-known version of the scattering

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theory, offered by Haag and Ruelle for the quantum field theory.[2] However, note that this method (as in the quantum field theory) generally does not lead to a complete decomposition of the Hilbert space into the direct sum of the constructed k-particle subspaces: the subspaces, corresponding to the "bound states" of our "particles", are left out.

We consider here the Ising model and the corresponding stochastic stationary Markov process, the so-called Glauber dynamics.[8] Spectral properties of the generator L of this dynamics for small values of the inverse temperature β were studied in ref. 2. It was shown that there are several invariant subspaces (one-, two-, ... k-particle subspaces), corresponding to the "lower", separate branches of the spectrum. The restriction of L to the one-particle subspace $\mathscr{H}_1$ is unitary equivalent to the operator of multiplication by an analytic function. The purpose of the present paper is to construct the invariant subspaces in which L represents the free k-particle dynamics, i.e., can be obtained by the second quantization of the restriction of L to the one-particle subspace.

The state space of the Ising model is the set

$$\Omega = \{-1, 1\}^{\mathbf{Z}^\nu}, \qquad \text{where} \quad \nu \in \mathbf{N}.$$

The formal Hamiltonian H of the model is

$$H(\sigma) = \beta \sum_{|x-y|=1} \sigma(x)\, \sigma(y), \qquad \sigma \in \Omega.$$

For sufficiently small β the Hamiltonian H determines a unique Gibbs measure μ_β on Ω (see ref. 9). There exists a reversible stationary Markov process $\{\eta_t, t \in \mathbf{R}\}$ with the space of states Ω and the stationary measure μ_β. The generator $L = L(\beta)$ of the corresponding stochastic semigroup is defined on the local functions $f: \Omega \to \mathbf{C}$ by

$$(Lf)(\sigma) = \sum_{x \in \mathbf{Z}^\nu} c_x(\sigma)[f(\sigma^x) - f(\sigma)]. \tag{1}$$

Here $\sigma^x(y) := -\sigma(y)$ if $x = y$ and $\sigma(y)$ otherwise. We assume that the functions $\{c_x(\cdot), x \in \mathbf{Z}^\nu\}$ are local and obey the following two properties:

1. The detail balance condition: for all $x \in \mathbf{Z}^\nu$ and $\sigma \in \Omega$

$$\frac{c_x(\sigma)}{c_x(\sigma^x)} = \exp[-(\Delta_x H)(\sigma)],$$

[2] It was pointed out in ref. 7 that the Haag–Ruelle theory can by applied to the spectral analysis of the generators of the stochastic dynamics.

where $(\Delta_x H)(\sigma) = H(\sigma^x) - H(\sigma) = -2\beta\sigma(x) \sum_{|x-y|=1} \sigma(y)$. It implies that the process is reversible and operator L can be extended by closure to a self-adjoint operator in the space $\mathscr{H} = L_2(\Omega, \mu_\beta)$ which we denote as before L.

2. The following representation is valid:

$$c_x(\sigma) = 1/2 + \sum_{B \subset Q+x} r_B^{(x)} \sigma_B.$$

Here

$$\sigma_B = \prod_{x \in B} \sigma(x), \tag{2}$$

Q is a fixed finite subset of $\mathbf{Z}^\nu$ and $Q+x$ is the shift of Q by the vector x. Coefficients $r_B^{(x)}$ are translation invariant:

$$r_B^{(x)} = r_{B+z}^{(x+z)}$$

and for some constant $K > 0$

$$|r_B^{(x)}| < K\beta.$$

Let U_s: $\mathscr{H} \to \mathscr{H}$, $s \in \mathbf{Z}^\nu$ be the representation of the translation group, given by

$$U_s f(x) = f(x-s).$$

We call an invariant with respect to L and $\{U_s\}$ subspace $\mathscr{H}_1 \subset \mathscr{H}$ *one-particle*, if it is cyclic with respect to $\{U_s\}$ and the spectrum of $L|_{\mathscr{H}_1}$ is separated from the rest of the spectrum of L. The following result of ref. 2 establishes the existence of the one-particle subspace $\mathscr{H}_1 \subset \mathscr{H}$ and yields a description of the restriction of L to $\mathscr{H}_1$:

Theorem 1. There exists $\beta_0 > 0$ such that for any β, $|\beta| < \beta_0$ there are three invariant with respect to L and $\{U_s\}$, mutually orthogonal subspaces $\mathscr{H}_0 \equiv \{const\}$, $\mathscr{H}_1$, $\mathscr{H}_{>1}$ such that

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_{>1}.$$

The spectrum of the restriction $L|_{\mathscr{H}_0}$ is $\{0\}$, the spectrum of $L_1 := L|_{\mathscr{H}_1}$ is contained in the interval $(-1-d_1, -1+d_1)$ and the spectrum of $L|_{\mathscr{H}_2}$ belongs to $(-\infty, -2+d_2)$. Here $d_i = d_i(\beta) > 0$ and $d_1 + d_2 < 1$, so that the spectra of $L|_{\mathscr{H}_i}$, $i = 0, 1, > 1$ do not intersect. Further, there exists a unitary operator $U: \mathscr{H}_1 \to L_2(T^\nu, dp)$ (T^ν is the ν-dimensional torus and dp is the normalized Haar measure) such that UL_1U^{-1} has the form

$$(UL_1U^{-1}f)(p) = f(p)\, m(p),$$

where the function $m(p)$ is analytic in a vicinity of T^ν and

$$\max_p |m(p)+1| = d_1 < c\beta$$

for some constant $c > 0$, independent of β. Moreover, there exist real functions $\{v_x\}$, $x \in \mathbf{Z}^\nu$, forming an orthonormal basis in $\mathscr{H}_1$. They admit the expansion

$$v_x = \sum_A K_{A,x} \sigma_A, \tag{3}$$

where σ_A are the monomials (2), $K_{A,x}$ are some coefficients and the sum is over finite subsets of $\mathbf{Z}^\nu$. Coefficients $K_{A,x}$ are translation invariant:

$$K_{A+z,\, x+z} = K_{A,x}$$

There exists $\lambda_1 = \lambda_1(\beta)$ such that $\lambda_1(\beta) \to 0$ as $\beta \to 0$ and

$$\sum_A |K_{A,x}|\, \lambda_1^{-d_{A \cup x}} = R < \infty. \tag{4}$$

Here $d_{A \cup x}$ stands for the minimal length of a connected graph containing $A \cup x$. The operator L acts on functions v_x as follows:

$$Lv_x = \sum_{y \in \mathbf{Z}^\nu} \hat{m}(y-x)\, v_x,$$

where

$$|\hat{m}(y)| \leqslant c\lambda_2^{|y|} \tag{5}$$

for some constants c and $\lambda_2 < 1$. Numbers $\hat{m}(y)$ are the Fourier coefficients of function $m(p)$.

Denote

$$\mathscr{H}^{(k)} = \bigotimes_1^k \mathscr{H}_1$$

and

$$L^{(k)} = L_1 \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} + \mathbf{1} \otimes L_1 \otimes \ldots \otimes \mathbf{1} + \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes L_1.$$

Let the Hilbert space $\mathscr{H}^{(k),\,symm}$ be the k-fold symmetric product of $\mathscr{H}_1$; $\mathscr{H}^{(k),\,symm}$ is an invariant subspace of $\mathscr{H}^{(k)}$ with respect to $L^{(k)}$. Let

$$\mathscr{F}^{symm}(\mathscr{H}_1) = \bigoplus_{k=0}^{\infty} \mathscr{H}^{(k),\,symm}$$

be the symmetric Fock space and

$$d\Gamma^{symm}(L_1) = \bigoplus_{k=0}^{\infty} L^{(k)} : \mathscr{F}^{symm}(\mathscr{H}_1) \to \mathscr{F}^{symm}(\mathscr{H}_1)$$

(second quantization of L_1).

In this paper we prove the following result:

Theorem 2. There exists $\beta_0 > 0$ such that for any β, $|\beta| < \beta_0$ there exists a subspace $\tilde{\mathscr{H}} \subset \mathscr{H}$ which is invariant with respect to L and for which the restriction $L|_{\tilde{\mathscr{H}}}$ is unitary equivalent to $d\Gamma^{symm}(L_1)$.

Note that by theorem 1 the operator $L^{(k)}$ is unitary equivalent to the operator of multiplication by the function

$$M_k(P^{(k)}) = \sum_{i=1}^{k} m(p_i), \qquad \text{where} \quad P^{(k)} = (p_1, \ldots, p_k) \in T^{kv}.$$

Then theorem 2 implies that the spectrum of L contains all the points which can be represented in the form $\sum_{i=1}^{k} m(p_i)$ for some k and p_i, $i = 1, \ldots, k$.

We will prove theorem 2 by constructing a wave operator. The proof will include the four stages:

1. For each nonnegative integer k we define a bounded inclusion operator J_k: $\mathscr{H}^{(k)} \to \mathscr{H}$.

2. We prove that there exists the wave operator

$$W_k = s - lim_{t \to +\infty} W_{k,\,t}, \tag{6}$$

where

$$W_{k,\,t} = \exp(-itL)\, J_k \exp(itL^{(k)}) \colon \mathscr{H}^{(k)} \to \mathscr{H}.$$

It follows that the restriction $L|_{\overline{\text{Ran}\, W_k}}$ is unitary equivalent to $L^{(k)}|_{\mathscr{H}^{(k)} \ominus \text{Ker}\, W_k}$ (see ref. 10).

3. We prove that $\text{Ker}\, W_k = \mathscr{H}^{(k)} \ominus \mathscr{H}^{(k),\,symm}$. Then restriction $L|_{\overline{\text{Ran}\, W_k}}$ is unitary equivalent to $L^{(k)}|_{\mathscr{H}^{(k),\,symm}}$.

4. We prove the orthogonality of $\overline{\operatorname{Ran} W_k}$ and $\overline{\operatorname{Ran} W_l}$ for $k \neq l$. It implies theorem 2 if we set $\tilde{\mathscr{H}} := \bigoplus_{k=0}^{\infty} \overline{\operatorname{Ran} W_k}$. For the next three sections, corresponding to the stages 1-3, we fix $k \geqslant 2$ (for $k = 0, 1$ the definition of J_k and the listed assertions are trivial). Index k is occasionally omitted in the notation $(J, W, W_t, M(P)$ instead of $J_k, W_k, W_{k,t}, M_k(P^{(k)}))$.

2. THE INCLUSION OPERATOR *J*.

Let $\{e_x\}_{x \in \mathbf{Z}^\nu}$ be the orthonormal basis in $\mathscr{H}_1$, formed by functions v_x, mentioned in theorem 1 (we introduce a different notation for the same objects to emphasize that they are considered here as vectors of a linear space).

Denote $X = (x_1, ..., x_k) \in \mathbf{Z}^{k\nu}$ and introduce in $\mathscr{H}^{(k)}$ an orthonormal basis

$$e_X = e_{x_1} \otimes e_{x_2} \otimes \dots \otimes e_{x_k}. \tag{7}$$

Consider the set

$$S = \{X \in \mathbf{Z}^{k\nu} \mid \min_{i \neq j} |x_i - x_j| \leqslant (\max_{i,j} |x_i - x_j|)^{1/2}\} \tag{8}$$

and define the inclusion operator J: $\mathscr{H}^{(k)} \to \mathscr{H}$ on the basis vectors as follows:

$$J(e_X) = \begin{cases} \prod_{i=1}^k v_{x_i} - \langle \prod_{i=1}^k v_{x_i} \rangle, & \text{if} \quad X \notin S \\ 0, & \text{if} \quad X \in S \end{cases}$$

where we denote $\langle \cdot \rangle = \int \cdot \, d\mu_\beta$. We will prove that thus defined J is bounded and therefore can be extended to all vectors of $\mathscr{H}^{(k)}$ by linearity and closure. The set S is introduced with the purpose of making J bounded. (Let us show that J', defined by $J'e_X = \prod_i v_{x_i} - \langle \prod_i v_{x_i} \rangle$ for all X, would be unbounded. The inner product in $\mathscr{H}$ has the form $(f, g) = \langle f \bar{g} \rangle$. It follows that

$$(J'e_X, J'e_Y) = \left\langle \prod_i v_{x_i} \prod_j v_{y_j} \right\rangle - \left\langle \prod_i v_{x_i} \right\rangle \left\langle \prod_j v_{y_j} \right\rangle.$$

Fix some $n < k$ and let $X_z = (x_1 + z, x_2 + z, ..., x_n + z, x_{n+1}, ..., x_k)$, $z \in \mathbf{Z}^\nu$. Then, as follows from the lemma proved below,

$$(J'e_{X_z}, J'e_Y) \to \left\langle \prod_{i=1}^n v_{x_i} \right\rangle \left(\left\langle \prod_{i=n+1}^k v_{x_i} \prod_{j=1}^k v_{y_j} \right\rangle - \left\langle \prod_{i=n+1}^k v_{x_i} \right\rangle \left\langle \prod_{j=1}^k v_{y_j} \right\rangle \right),$$

as $z \to \infty$. The limit is generally nonzero, though vectors e_{X_z} are orthonormal. This shows that J' cannot be bounded.)

Lemma 1. Let $a_1, \ldots, a_s, b_1, \ldots, b_t \in \mathbf{Z}^\nu$. There exist constants $\lambda < 1$ and $\tilde{c}_{s,t}$ such that

$$\left| \left\langle \prod_{i=1}^{s} v_{a_i} \prod_{j=1}^{t} v_{b_j} \right\rangle - \left\langle \prod_{i=1}^{s} v_{a_i} \right\rangle \left\langle \prod_{j=1}^{t} v_{b_j} \right\rangle \right| < \tilde{c}_{s,t} \lambda^{\rho(\{a\}_i, \{b\}_j)}, \tag{9}$$

where $\rho(\cdot, \cdot)$ is the distance between sets.

Proof. By (4), the l.h.s. of (9) is majorized by

$$\sum_{\substack{A_i,\, i=1, \ldots, s\\ B_j,\, j=1, \ldots, t}} \prod_{i=1}^{s} |K_{A_i, a_i}| \cdot \prod_{j=1}^{t} |K_{B_j, b_j}| \cdot |\langle \sigma_{\Delta\{A_i\}} \cdot \sigma_{\Delta\{B_j\}} \rangle - \langle \sigma_{\Delta\{A_i\}} \rangle \langle \sigma_{\Delta\{B_j\}} \rangle|,$$

where

$$\Delta\{A_i\} := \{x \in \mathbf{Z}^\nu \mid x \text{ belongs to an odd number of } A_i\} \tag{10}$$

and similarly for $\Delta\{B_j\}$. Divide this sum into two: $\sum_1 + \sum_2$; $\sum_1$ is over those A_i, B_j, for which $d_{a_i \cup A_i} < \rho(\{a_i\}_1^s, \{b_j\}_1^t)/3$, $d_{b_j \cup B_j} < \rho(\{a_i\}_1^s, \{b_j\}_1^t)/3$; while $\sum_2$ is over the rest of A_i, B_j. Let us first estimate $\sum_1$. Let us use the following decay of correlations property (see ref. 9):

$$|\langle \sigma_A \cdot \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle| \leqslant c_1^{d_A + d_B} \lambda_2^{\rho(A, B)},$$

where c_1 and $\lambda_2 < 1$ are some constants. Set $A := \Delta\{A_i\}$, $B := \Delta\{B_j\}$ and note that

$$\rho(A, B) \geqslant \rho(\{a_i\}_1^s, \{b_j\}_1^t) - \max_i \{d_{a_i \cup A_i}\} - \max_j \{d_{b_j \cup B_j}\} \geqslant \rho(\{a_i\}_1^s, \{b_j\}_1^t)/3.$$

Then we have

$$\sum\nolimits_1 \leqslant \sum_{A_i, B_j} \prod_i |K_{A_i, a_i}| \cdot \prod_j |K_{B_j, b_j}| \cdot c_1^{d_A + d_B} \lambda_2^{\rho(\{a_i\}_1^s, \{b_j\}_1^t)/3}. \tag{11}$$

Note that $d_A + d_B \leqslant \sum_i d_{a_i \cup A_i} + \sum_j d_{b_j \cup B_j}$; λ_1, appearing in (4), is less than $1/c_1$ for sufficiently small β, therefore

$$c_1^{d_A + d_B} \leqslant \lambda_1^{-\sum_i d_{a_i \cup A_i} - \sum_j d_{b_j \cup B_j}}.$$

Taking this inequality to (11) and using (4), we find that

$$\sum_1 \leqslant R^{s+t} \lambda_2^{\rho(\{a_i\}_1^s,\, \{b_j\}_1^t)/3}.$$

Now let us estimate $\sum_2$; note that $|\langle \sigma_{\Delta\{A_i\}} \cdot \sigma_{\Delta\{B_j\}} \rangle - \langle \sigma_{\Delta\{A_i\}} \rangle \langle \sigma_{\Delta\{B_j\}} \rangle| \leqslant 2$. Furthermore, for A_i, B_j, corresponding to $\sum_2$, the following inequality holds:

$$-\sum_i d_{a_i \cup A_i} - \sum_j d_{b_j \cup B_j} + \rho(\{a_i\}_1^s,\, \{b_j\}_1^t)/3 \leqslant 0.$$

Denote the l.h.s. of this inequality by α and note that $\lambda_1^{\alpha} > 1$ as long as $\lambda_1 < 1$. That yields

$$\sum_2 \leqslant \lambda_1^{\alpha} \sum_2 \leqslant 2 \sum_{A_i,\, B_j} \prod_i |K_{A_i,\, a_i}| \cdot \prod_i |K_{B_j,\, b_j}| \cdot \lambda_1^{\alpha}.$$

Now by (4) we obtain

$$\sum_2 \leqslant 2R^{s+t} \lambda_1^{\rho(\{a_i\}_1^s,\, \{b_j\}_1^t)/3}.$$

Thus we finally find that

$$\sum = \sum_1 + \sum_2 \leqslant \tilde{c}_{s,\, t} \lambda^{\rho(\{a_i\}_1^s,\, \{b_j\}_1^t)},$$

where $\tilde{c}_{s,\, t} = 3R^{s+t}$, $\lambda = (\max(\lambda_1,\, \lambda_2))^{1/3}$. ∎

We prove the boundedness of J by showing that

$$\max_{X \in \mathbf{Z}^{kv}} \sum_{Y \in \mathbf{Z}^{kv}} |(Je_X,\, Je_Y)| < \infty.$$

(Indeed, if $f = \sum_X c_X e_X$, then

$$(Jf, Jf) = \sum_{X,\, Y} c_X \overline{c_Y} (Je_X,\, Je_Y) \leqslant \sum_{X,\, Y} \frac{|c_X|^2 + |c_Y|^2}{2} |(Je_X,\, Je_Y)|$$

$$\leqslant \sum_X |c_X|^2 \max_X \sum_Y |(Je_X,\, Je_Y)| = \|f\|^2 \max_X \sum_Y |(Je_X,\, Je_Y)|).$$

Let us call sequences $X=(x_1, ..., x_k)$ and $Y=(y_1, ..., y_k)$ equivalent ($X \sim Y$), if for some permutation π $x_n = y_{\pi(n)}$, $n=1, ..., k$. Note that for any X the number of Y's, equivalent to the X, does not exceed $k!$. Furthermore, note that (4) implies $|\sup_\sigma v_x(\sigma)| \leqslant R$. Therefore,

$$\max_X \sum_{Y \sim X} |(Je_X, Je_Y)| \leqslant k! \max_X |(Je_X, Je_Y)| \leqslant k! \max_X \left| \left\langle \prod_{i=1}^k v_{x_i}^2 \right\rangle - \left\langle \prod_{i=1}^k v_{x_i} \right\rangle^2 \right| \leqslant 2k! R^{2k} < \infty.$$

Thus, it remains to prove that

$$\max_X \sum_{Y \nsim X} |(Je_X, Je_Y)| < \infty. \tag{12}$$

Since $(Je_{X+z}, Je_{Y+z}) = (Je_X, Je_Y)$, $z \in \mathbf{Z}^\nu$, where we denote $X+z := (x_1+z, x_2+z, ..., x_k+z)$ and similarly for Y, then we can assume without loss of generality that in (12) $x_1 = 0$. Therefore (12) will follow from the finiteness of the sum

$$\sum_{X : x_1 = 0} \sum_{Y \nsim X} |(Je_X, Je_Y)|. \tag{13}$$

Consider a set of pairs

$$D_n = \{(X, Y) \in \mathbf{Z}^{k\nu} \times \mathbf{Z}^{k\nu} \mid \max_i (|x_i|, |y_i|) \in [n, n+1), x_1 = 0, \\ Y \nsim X, X \notin S, Y \notin S\},$$

where S is defined in (8). Since $(Je_X, Je_Y) = 0$ if $X \in S$ or $Y \in S$, rewrite (13) as

$$\sum_{n=1}^{\infty} \sum_{(X, Y) \in D_n} |(Je_X, Je_Y)|. \tag{14}$$

Lemma 2. Let $(X, Y) \in D_n$. Then

$$|(Je_X, Je_Y)| \leqslant c\lambda^{\sqrt{n}/6} \tag{15}$$

for some constant c, independent of n.

Proof. Consider the four possible cases.

1. $\max|x_i| < n/3$, $\min|y_j| \geqslant 2n/3$

In this case $\rho(\{x_i\}_1^s, \{y_j\}_1^t) \geqslant n/3$ and by lemma 1 $|(Je_X, Je_Y)| = |\langle \prod_i v_{x_i} \prod_j v_{y_j}\rangle - \langle \prod_i v_{x_i}\rangle\langle \prod_j v_{y_j}\rangle| \leqslant \tilde{c}_{k,k}\lambda^{n/3}$.

2. $\max|x_i| < n/3$, $\min|y_j| < 2n/3$

In this case it follows from the definition of D_n that $\max|y_j| \in [n, n+1)$, therefore $\max_{i,j}|y_i - y_j| \geqslant n/3$ and, since $Y \notin S$, then $\min_{i\neq j}|y_i - y_j| \geqslant \sqrt{n/3}$. Write

$$|(Je_X, Je_Y)| \leqslant \left|\left\langle \prod_{i=1}^k v_{x_i} \prod_{j=1}^k v_{y_j}\right\rangle\right| + \left|\left\langle \prod_{i=1}^k v_{x_i}\right\rangle\right|\left|\left\langle \prod_{j=1}^k v_{y_j}\right\rangle\right| \tag{16}$$

Let $|y_{j_0}| \in [n, n+1)$. Then $\rho(y_{j_0}, \{y_j\}_{j\neq j_0}) \geqslant \sqrt{n/3}$, $\rho(y_{j_0}, \{x_i\}_1^k) \geqslant 2n/3 \geqslant \sqrt{n/3}$. Let us apply lemma 1 in the following way: set $s=1$ and $t=2k-1$, choose $a_1 = y_{j_0}$ and take the rest of y_j's and all of x_j's as $b_1, \ldots, b_{2k-1}$. Then, since $\langle v_{y_{j_0}}\rangle = 0$ (see ref. 2), we have

$$\left\langle \prod_{i=1}^k v_{x_i} \prod_{j=1}^k v_{y_j}\right\rangle\Bigg| = \left|\left\langle \prod_{i=1}^k v_{x_i} \prod_{j=1}^k v_{y_j}\right\rangle - \left\langle \prod_{i=1}^k v_{x_i} \prod_{j\neq j_0}^k v_{y_j}\right\rangle\right| \langle v_{y_{j_0}}\rangle| \leqslant \tilde{c}_{1,2k-1}\lambda^{\sqrt{n/3}}.$$

In order to estimate the second summand in the r.h.s. of (16), write the bound $|\langle \prod_i v_{x_i}\rangle| \leqslant |\sup_\sigma v_{x_i}(\sigma)|^k \leqslant R^k$ for the first factor. Further, let us apply lemma 1 to v_{y_j}, setting $s=1$, $t=k-1$ and choosing $a_1 = y_{j_0}$ and the rest of y_j's as $b_1, \ldots, b_{k-1}$. Then

$$\left|\left\langle \prod_{j=1}^k v_{y_j}\right\rangle\right| = \left|\left\langle \prod_{j=1}^k v_{y_j}\right\rangle - \left\langle \prod_{j\neq j_0} v_{y_j}\right\rangle \langle v_{y_{j_0}}\rangle\right| \leqslant \tilde{c}_{1,k-1}\lambda^{\sqrt{n/3}}.$$

We finally obtain in this case that $|(Je_X, Je_Y)| \leqslant (\tilde{c}_{1,2k-1} + R^k\tilde{c}_{1,k-1})\lambda^{\sqrt{n/3}}$.

3. $\max|x_i| \geqslant n/3$ and for some i_0 $\rho(x_{i_0}, \{y_j\}_1^k) \geqslant \sqrt{n}/6$.

Note that $\max_{i,j}|x_i - x_j| \geqslant n/3$, because $x_1 = 0$ and $\max|x_i| \geqslant n/3$. Hence $\min_{i\neq j}|x_i - x_j| \geqslant \sqrt{n/3}$ by definition of S. In particular, $\rho(x_{i_0}, \{x_i\}_{i\neq i_0}) \geqslant \sqrt{n/3} \geqslant \sqrt{n}/6$ and $\rho(x_{i_0}, \{x_i\}_{i\neq i_0} \cup \{y_j\}_1^k) \geqslant \sqrt{n}/6$. Applying again lemma 1 to partitions $\{x_i\}_1^k = \{x_{i_0}\} \cup \{x_i\}_{i\neq i_0}$ and $\{x_i\}_1^k \cup \{y_j\}_1^k = \{x_{i_0}\} \cup (\{x_i\}_{i\neq i_0} \cup \{y_j\}_1^k)$, we obtain $|(Je_X, Je_Y)| \leqslant (\tilde{c}_{1,2k-1} + R^k\tilde{c}_{1,k-1})\lambda^{\sqrt{n}/6}$.

4. $\max|x_i| \geqslant n/3$ and for all i $\rho(x_i, \{y_j\}_1^k) < \sqrt{n}/6$.

Here we have again $\max_{i,j}|x_i - x_j| \geqslant n/3$ and $\min_{i\neq j}|x_i - x_j| \geqslant \sqrt{n/3}$. Further, $\max_{i,j}|y_i - y_j| \geqslant \max_{i,j}|x_i - x_j| - 2\max_i \rho(x_i, \{y_j\}_1^k) \geqslant n/3 - 2\sqrt{n}/6 \geqslant n/9$ for sufficiently large n; therefore $\min_{i\neq j}|y_i - y_j| \geqslant \sqrt{n}/3$, because $Y \notin S$. Since $X \nsim Y$, one can find i_0 such that $\rho(x_{i_0}, \{y_j\}_1^k) > 0$. As by assumption for all i $\rho(x_i, \{y_j\}_1^k) < \sqrt{n}/6$, then one can find j_0 such that $|x_{i_0} - y_{j_0}| < \sqrt{n}/6$ and $x_{i_0} \neq y_{j_0}$. Since $\min_{i \neq j} |y_i - y_j| \geqslant \sqrt{n}/3$ and $\min_{i \neq j} |x_i - x_j| \geqslant \sqrt{n/3} > \sqrt{n}/3$, then $\rho(\{x_{i_0}\} \cup \{y_{j_0}\}, \{x_i\}_{i \neq i_0} \cup \{y_j\}_{j \neq j_0}) \geqslant \min(\min_{i \neq j} |x_i - x_j|, \min_{i \neq j} |y_i - y_j|) - |x_{i_0} - y_{j_0}| \geqslant \sqrt{n}/3 - \sqrt{n}/6 = \sqrt{n}/6$. Using $\langle v_{x_{i_0}} \rangle = 0$ and $\langle v_{x_{i_0}} v_{y_{j_0}} \rangle = 0$ (orthogonality), write

$$\begin{aligned}
|(Je_X, Je_Y)| &\leqslant \left| \left\langle \prod_{i=1}^k v_{x_i} \prod_{j=1}^k v_{y_j} \right\rangle \right| + \left| \left\langle \prod_{i=1}^k v_{x_i} \right\rangle \left\langle \prod_{j=1}^k v_{y_j} \right\rangle \right| \\
&= \left| \left\langle \prod_{i=1}^k v_{x_i} \prod_{j=1}^k v_{y_j} \right\rangle - \langle v_{x_{i_0}} v_{y_{j_0}} \rangle \left\langle \prod_{i \neq i_0} v_{x_i} \prod_{j \neq j_0} v_{y_j} \right\rangle \right| \\
&\quad + \left| \left\langle \prod_{i=1}^k v_{x_i} \right\rangle - \left\langle \prod_{i \neq i_0} v_{x_i} \right\rangle \langle v_{x_{i_0}} \rangle \right| \cdot \left| \left\langle \prod_{j \neq j_0} v_{y_j} \right\rangle \right|.
\end{aligned} \tag{17}$$

Then lemma 1 and inequalities $\rho(\{x_{i_0}\} \cup \{y_{j_0}\}, \{x_i\}_{i \neq i_0} \cup \{y_j\}_{j \neq j_0}) \geqslant \sqrt{n}/6$ and $\rho(\{x_{i_0}\}, \{x_i\}_{i \neq i_0}) \geqslant \sqrt{n}/6$ imply that the first summand in the r.h.s. of (17) is bounded by $\tilde{c}_{2, 2k-2} \lambda^{\sqrt{n}/6}$, while the second one is bounded by $R^k \tilde{c}_{1, k-1} \lambda^{\sqrt{n}/6}$, i.e., $|(Je_X, Je_Y)| \leqslant (\tilde{c}_{2, 2k-2} + R^k \tilde{c}_{1, k-1}) \lambda^{\sqrt{n}/6}$ for sufficiently large n.

So, if the constant c is chosen large enough, then the bound (15) holds true in all of the four cases considered above. ▮

Note that the cardinality $|D_n| \leqslant (n+1)^{2kv}$. Then by lemma 2 the series (14) is majorized by the convergent series $\sum_{n=1}^{\infty} (n+1)^{2kv} c \lambda^{\sqrt{n}/6}$ and, therefore, converges. The boundedness of J is proven.

3. THE EXISTENCE OF THE WAVE OPERATOR *W*.

By the Cook method, the existence of the wave operator (6) will be proved if we show that for a dense set $\{u\} \subset \mathscr{H}^{(k)}$

$$\int_0^{+\infty} \|(LJ - JL^{(k)}) \exp(itL^{(k)}) u\| \, dt < \infty \tag{18}$$

Denote $u_t := \exp(itL^{(k)}) u \in \mathscr{H}^{(k)}$. Let $u_{t, X} := (u_t, e_X) \in \mathbf{C}$ be the coefficients in the expansion

$$u_t = \sum_{X \in \mathbf{Z}^{kv}} u_{t, X} e_X,$$

where $\{e_X\}_{X \in \mathbf{Z}^{k\nu}}$ is the orthonormal basis introduced in (7). Let $T^{k\nu} = [-\pi, \pi)^{k\nu}$ be the $k\nu$-dimensional torus and dP be the normalized Haar measure on it. Define a unitary isomorphism $F: \mathscr{H}^{(k)} \to L_2(T^{k\nu}, dP)$ by

$$(Fu)(P) = \sum_{X \in \mathbf{Z}^{k\nu}} (u, e_X) \exp(i(P, X)),$$

where (u, e_X) and (P, X) are the inner products in $\mathscr{H}^{(k)}$ and $\mathbf{R}^{k\nu}$. Note that $u_{t,X}$, $X \in \mathbf{Z}^{k\nu}$ are the Fourier coefficients of Fu_t. Denote $\hat{u} := Fu$, $\hat{u}_t := Fu$. By Theorem 1, $FL^{(k)}F$ is the operator of multiplication by the analytic function

$$M(P) = \sum_{n=1}^{k} m(p_n), \qquad \text{where} \quad P = (p_1, \ldots, p_k),\ p_n \in T^{\nu}.$$

Therefore

$$u_{t,X} = \int_{T^{k\nu}} \exp(i[tM(P) - (P, X)])\, \hat{u}(P)\, dP. \tag{19}$$

The integral (18) is upper bounded by

$$\sum_{X \in \mathbf{Z}^{k\nu}} \|(LJ - JL^{(k)}\| \int_0^{+\infty} |u_{t,X}|\, dt. \tag{20}$$

We will prove that the sum converges; the proof will be carried out in the following way. We divide the sum into two parts. The first part includes the summands for which the integral $\int_0^{+\infty} |u_{t,X}|\, dt$ is small. By the stationary phase method, this is the case if in (19) the gradient $\nabla_P[tM(P) - (P, X)] \neq 0$, i.e., $X \neq t\nabla_P M(P)$ for all $P \in supp\, \hat{u}$. Using the boundedness of the second factor $\|(LJ - JL^{(k)})\, e_X\|$, we prove the convergence of the first sum. The finiteness of the second sum follows from the smallness of $\|(LJ - JL^{(k)})\, e_X\|$. We show that this quantity exponentially decreases as $\min_{i \neq j} |x_i - x_j| \to \infty$, where $(x_1, \ldots, x_k) = X$. The second sum includes (roughly speaking) those summands, for which $X/t \in \{\nabla_P M(P) \mid P \in supp\, \hat{u}\}$ for some t, therefore $\hat{u}$ is chosen so that

$$\{\nabla_P M(P) \mid P \in supp\, \hat{u}\} \cap \{(x_1, \ldots, x_k) \in \mathbf{R}^{k\nu} | \exists i \neq j : x_i = x_j\} = \varnothing. \tag{21}$$

Analyticity of $M(P)$ implies that there is a dense set of such $\hat{u}$.

Let us proceed to the precise formulation. Consider the set $T_0 \subset T^{k\nu}$:

$$T_0 = \{P \in T^{k\nu} \mid \exists i \neq j : \nabla_{p_i} M(P) = \nabla_{p_j} M(P)\},$$

where ∇_{p_i} are the gradients with respect to the corresponding coordinates. Let function $\hat{u} \in C^\infty(T^{kv})$ is such that $supp\, \hat{u} \cap T_0 \neq \emptyset$ (which is equivalent to (21)). The set T_0 is closed and has zero measure by the analyticity of $M(P)$, therefore such functions are dense in $L_2(T^{kv})$. Let Ω be an open bounded subset of $\mathbf{R}^{kv}$, containing the set

$$\{\nabla_P M(P) \mid P \in supp\, \hat{u}\} \tag{22}$$

and not intersecting $\{(x_1, ..., x_k) \in \mathbf{R}^{kv} \mid \exists i \neq j : x_i = x_j\}$. Such Ω exists according to the choice of $\hat{u}$ and the boundedness of the set (22). The following estimate holds (see ref. 10):

Lemma 3 (stationary phase method). For any $n \in \mathbf{R}$ there exists a constant c such that $|u_{t,X}| \leqslant c(1+|X|+|t|)^{-n}$ for all $X \in \mathbf{Z}^{kv}$, $t \in \mathbf{R}$, for which $X \notin t\Omega$ (we denote here by $|X|$ the length of the vector X and $t\Omega := \{t\omega | \omega \in \Omega\}$).

Let

$$Z' := \{X \in \mathbf{Z}^{kv} \mid \forall t > 0\ X \notin t\Omega\},$$
$$Z'' := Z^{kv} \backslash Z' = \{X \in \mathbf{Z}^{kv} \mid \exists t > 0 \quad X \in t\Omega\}.$$

Then the sum (20) can be rewritten as

$$\sum_{X \in \mathbf{Z}^{kv}} \|(LJ - JL^{(k)}\| \int_0^{+\infty} |u_{t,X}|\, dt = \sum_{X \in Z'} + \sum_{X \in Z''}. \tag{23}$$

By lemma 2,

$$\int_0^{+\infty} |u_{t,X}|\, dt \leqslant c \int_0^{+\infty} (1+|X|+t)^{-n}\, dt = \frac{c}{n-1} (1+|X|)^{-n+1}$$

if $X \in Z'$. Let us choose n sufficiently large, so that the series $\sum_{X \in Z'} (1+|X|)^{-n+1}$ converges. It follows that the first summand in the r.h.s. of (23) is finite, if we prove that $\sup_X \|(LJ - JL^{(k)})\, e_X\| < \infty$. Since $L^{(k)}$ and J are bounded, it suffices to prove that $\sup_X \|LJe_X\| < \infty$. Since $Je_X = 0$ for $X \in S$, we assume without loss of generality that $X \notin S$. Note that by (1) $L\sigma_A = -2 \sum_{x \in A} c_x \sigma_A$. The norm $\|c_x\|$ is finite and independent of x; hence $\|L\sigma_A\| \leqslant 2\, |A|\, \|c_x\|$. Using the expansion (3) and the notation (10), we write:

$$\|LJe_X\| \leqslant \left\| L\left(\prod_{l=1}^{k} v_{x_l}\right)\right\| = \left\| \sum_{A_1, \ldots, A_k} \prod_{l=1}^{k} K_{A_l, x_l} L(\sigma_{\Delta\{A_l\}})\right\|$$
$$\leqslant 2 \sum_{A_1, \ldots, A_k} \prod_{l=1}^{k} |K_{A_l, x_l}| \, \|c_x\| \, |\Delta\{A_l\}|$$
$$\leqslant 2 \, \|c_x\| \prod_{l=1}^{k} \left(\sum_{A_l} |K_{A_l, x_l}|(1+|A_l|) \right) < \infty$$

uniformly in X due to (4).

So, it remains to prove that $\sum_{X \in Z''} < \infty$. Let us first estimate the integral $\int_0^{+\infty} |u_{t,X}| \, dt$ for $X \in Z''$. Let $c_1 = \inf_{X \in \Omega} |X|$. By definition of Ω, $c_1 > 0$. Represent the integral as the sum

$$\int_0^{+\infty} |u_{t,X}| \, dt = \int_0^{|X|/c_1} + \int_{|X|/c_1}^{+\infty}.$$

Since $|u_{t,X}| \leqslant \|\hat{u}\|$, the first summand in the r.h.s. is not greater than $\|\hat{u}\| \, |X|/c_1$. Further, by definition of c_1, we have $X \notin t\Omega$ for $t > |X|/c_1$. Hence, by lemma 3,

$$\int_{|X|/c_1}^{+\infty} |u_{t,X}| \, dt \leqslant c \int_0^{+\infty} (1+|X|+t)^{-2} \leqslant c.$$

Thus,

$$\int_0^{+\infty} |u_{t,X}| \, dt \leqslant c + \frac{\|\hat{u}\|}{c_1} |X|.$$

It follows that the convergence of $\sum_{X \in Z''}$ in (23) will be established if we prove that $\|(LJ - JL^{(k)}) \, e_X\| \to 0$ sufficiently fast as $|X| \to \infty$, $X \in Z''$, so that

$$\sum_{X \in Z''} \left(c + \frac{\|\hat{u}\|}{c_1} |X| \right) \|(LJ - JL^{(k)}) \, e_X\| < \infty. \tag{24}$$

Note that if $X \in Z''$ and $|X|$ is sufficiently large, then $X \notin S$. Indeed, let

$$\begin{aligned} c_2 &:= \inf_{i \neq j, \, X \in \Omega} |x_i - x_j|, \\ c_3 &:= \sup_{X \in \Omega} |X|. \end{aligned} \tag{25}$$

By the choice of Ω we have $c_2 > 0$ and $c_3 < \infty$.

If $X \in Z''$, then for some t $X/t \in \Omega$, therefore $\min_{i \neq j} |x_i - x_j| \geqslant c_2 t$, $|X| < c_3 t$. It follows that

$$\min_{i \neq j} |x_i - x_j| \geqslant \frac{c_2}{c_3} |X|. \tag{26}$$

Since $\max_{i,j} |x_i - x_j| \leqslant \max |x_i| \leqslant |X|$, then for $|X| > 2c_3^2/c_2^2$ we have $\min_{i \neq j} |x_i - x_j| > (\max_{i,j} |x_i - x_j|)^{1/2}$, i.e. $X \notin S$. So, assume without loss of generality that $X \notin S$. Represent $(LJ - JL^{(k)})\, e_X$ as the sum $I_1(X) + I_2(X)$, where

$$I_1(X) = LJe_X - \sum_{i=1}^{k} \left(\prod_{j \neq i} v_{x_j} \right) L v_{x_i},$$

$$I_2(X) = \sum_{i=1}^{k} \left(\prod_{j \neq i} v_{x_j} \right) L v_{x_i} - JL^{(k)} e_X,$$

and estimate $I_1(X)$ and $I_2(X)$ separately. Using the expansion (3), write

$$I_1(X) = \sum_{A_1, \ldots, A_k} \prod_{i=1}^{k} K_{A_i, x_i} \left[L(\sigma_{\Delta\{A_i\}}) - \sum_{i=1}^{k} \left(\prod_{j \neq i} \sigma_{A_j} \right) L \sigma_{A_i} \right]. \tag{27}$$

Note that if $A_1, \ldots, A_k$ do not intersect, then $\Delta\{A_i\} = A_1 \amalg A_2 \amalg \ldots \amalg A_k$ and $L(\sigma_{\Delta\{A_i\}}) = \sum_{i=1}^{k} (\prod_{j \neq i} \sigma_{A_j})\, L\sigma_{A_i}$, so that the summands in (27), for which $A_1, \ldots, A_k$ do not intersect, equal 0. Denote by $\sum'$ the sum of those summands, where at least two of the sets $A_1, \ldots, A_k$ intersect. Then

$$\|I_1(X)\| \leqslant 4 \sum{}' \left(\prod_{i=1}^{k} |K_{A_i, x_i}| \right) \sum_{i=1}^{k} |A_i|\, \|c_x\| \leqslant 4 \sum{}' \left[\prod_{i=1}^{k} |K_{A_i, x_i}|\, (1 + |A_i|) \right] \|c_x\|. \tag{28}$$

Note that for fixed x and $\epsilon > 0$

$$(1 + |A|) \left(\frac{\lambda_1 + \epsilon}{\lambda_1} \right)^{-d_{x \cup A}} < 1$$

for all A, except for a finite number. It follows from here and from (4) that

$$\sum_{A} |K_{A < x}|\, (1 + |A|)(\lambda_1 + \epsilon)^{-d_{x \cup A}} =: R_1 < \infty.$$

For intersecting $A_1, \ldots, A_k$ we have inequality

$$\sum_{i=1}^{k} d_{x_i \cup A_i} - \min_{i \neq j} |x_i - x_j| \geqslant 0. \tag{29}$$

We have $\lambda_1 < 1$; choose ϵ small enough, so that $\lambda_1 + \epsilon < 1$. Denote by α the l.h.s. of (29), then $(\lambda_1+\epsilon)^{-\alpha} > 1$. Multiplication of the r.h.s. of (28) by $(\lambda_1+\epsilon)^{-\alpha}$ then yields

$$\|I_1(X)\| \leqslant 4 \sum{}' \left[\prod_{i=1}^{k} |K_{A_i, x_i}| \, (1+|A_i|)(\lambda_1+\epsilon)^{-d_{x_i \cup A_i}} \right] \|c_x\| \, (\lambda_1+\epsilon)^{\min_{i\neq j}|x_i - x_j|}$$
$$\leqslant 4 \, \|c_x\| \, R_1^k (\lambda_1+\epsilon)^{\min_{i \neq j}|x_i - x_j|}.$$

As was shown above, $\min_{i \neq j} |x_i - x_j| \geqslant c_2 \, |X|/c_3$, which gives us the desired bound

$$\|I_1(X)\| \leqslant 4 \, \|c_x\| \, R_1^k (\lambda_1+\epsilon)^{c_2|X|/c_3}.$$

Now let us estimate $\|I_2(X)\|$. Since $Lv_x = \sum_y \hat{m}(y-x) \, v_y$,

$$\sum_{i=1}^{k} \left(\prod_{j \neq i} v_{x_j} \right) Lv_{x_i} = \sum_{i=1}^{k} \sum_{y \in \mathbf{Z}^\nu} \left(\prod_{j \neq i} v_{x_j} \right) \hat{m}(y - x_i) \, v_y.$$

Recall the bound (5) and cast out the summands corresponding to those i's and y's, for which $|y - x_i| > a$, where $a := \min_{i \neq j} |x_i - x_j|/2$. By (26), $a \geqslant c_2 \, |X|/2c_3$. Note that for any $\epsilon > 0$

$$\sum_{z \in \mathbf{Z}^\nu, \, |z| > a} \lambda_2^{|z|} = o((\lambda_2+\epsilon)^a) \qquad \text{as} \quad a \to +\infty,$$

hence the part of the sum we cast out is $o((\lambda_2+\epsilon)^{c_2 \, |X|/2c_3})$ as $X \to \infty$. Denote $X_i[y] = (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)$ and note that if $x \in Z''$, $|y - x_i| < \min_{i \neq j} |x_i - x_j|/2$ and $|X|$ is sufficiently large, then $X_i[y] \in S$.

By definition of J and $L^{(k)}$,

$$JL^{(k)} e_X = \sum_{i=1}^{k} \sum_{y \in \mathbf{Z}^\nu, \, X_i[y] \in S} \hat{m}(y - x_i) \left(v_y \prod_{j \neq i} v_{x_j} - \left\langle v_y \prod_{j \neq i} v_{x_i} \right\rangle \right).$$

In this sum we also cast out the summands for which $|y - x_i| > a$, so that the part we cast out is $o((\lambda_2+\epsilon)^{c_2 \, |X|/2c_3})$. Then

$$\|I_2(x)\| = \left\| \sum_{i=1}^{k} \left(\prod_{j \neq i} v_{x_j} \right) Lv_{x_i} - JL^{(k)} e_X \right\|$$
$$\leqslant \sum_{i=1}^{k} \sum_{y \in \mathbf{Z}^\nu : \, |y - x_i| \leqslant a} |\hat{m}(y - x_i)| \cdot \left| \left\langle v_y \prod_{j \neq i} v_{x_j} \right\rangle \right| + o((\lambda_2+\epsilon)^{c_2 \, |X|/2c_3}).$$

If $y-x_i| \leqslant \min_{i\neq j}|x_i-x_j|/2$, then $\rho(y, \{x_j\}_{j\neq i}) \geqslant \min_{i\neq j}|x_i-x_j|/2$, hence, by lemma 1,

$$\left|\left\langle v_y \prod_{j\neq i} v_{x_j}\right\rangle\right| \leqslant \tilde{c}_{1,k-1}\lambda^{\min|x_i-x_j|/2} \leqslant \tilde{c}_{1,k-1}\lambda^{c_2|X|/2c_3}$$

by inequality (26) for $X \in Z''$. Therefore,

$$\|I_2(X)\| = O(\lambda^{c_2|X|/2c_3}) + o((\lambda_2+\epsilon)^{c_2|X|/2c_3}).$$

The exponential decrease of $\|I_1(X)\|$ and $\|I_2(X)\|$ implies (24). The existence of the wave operator W is proven.

4. Ker $W = \mathscr{H}^{(k)} \ominus \mathscr{H}^{(k),\,symm}$

The inclusion $\operatorname{Ker} W \supset \mathscr{H}^{(k)} \ominus \mathscr{H}^{(k),\,symm}$ is trivial. The opposite inclusion will be proven if we establish that for a dense subset $\{u\} \subset \mathscr{H}^{(k),\,symm}$

$$\|Wu\|^2 = k!\,\|u\|^2. \tag{30}$$

As before, suppose that $\hat{u} \in C^\infty(T^{kv})$, $\nabla_{p_i}M(P) \neq \nabla_{p_j}M(P)$ for $P \in supp\,\hat{u}$ and Ω is an open set, corresponding to $\hat{u}$. Ω can be chosen symmetric with respect to the planes $p_i = p_j$. Lemma 3 implies that

$$u_t = \sum_{X \in t\Omega \cap \mathbf{Z}^{kv}} u_{t,X} e_X + o(1), \qquad t \to +\infty$$

and

$$\lim_{t\to+\infty} \sum_{X \in t\Omega \cap \mathbf{Z}^{kv}} |u_{t,X}|^2 = \|u\|^2.$$

Respectively, by the boundedness of J,

$$\|Wu\|^2 = \lim_{t\to+\infty} \|W_t u\|^2 = \lim_{t\to+\infty} \|Ju_t\|^2$$

$$= \lim_{t\to+\infty} \left[\sum_{X,Y \in t\Omega \cap \mathbf{Z}^{kv}} (Je_X, Je_Y)\, u_{t,X} \overline{u_{t,Y}}\right].$$

If $X, Y \in t\Omega \cap \mathbf{Z}^{kv}$ and t is sufficiently large, then $X \notin S$, $Y \notin S$; hence

$$(Je_X, Je_Y) = \left\langle \prod_{i=1}^k v_{x_i} \prod_{j=1}^k v_{y_j}\right\rangle - \left\langle \prod_{i=1}^k v_{x_i}\right\rangle\left\langle \prod_{j=1}^k v_{y_j}\right\rangle.$$

By (25), $\min_{i\neq j}|x_i-x_j|\geqslant c_2t$ for $X\in t\Omega$, therefore, by lemma 1,

$$\left|\left\langle\prod_{i=1}^{k}v_{x_i}\right\rangle\right|=\left|\left\langle\prod_{i=1}^{k}v_{x_i}\right\rangle-\langle v_{x_1}\rangle\left\langle\prod_{i=2}^{k}v_{x_i}\right\rangle\right|\leqslant\tilde{c}_{1,\,k-1}\lambda^{c_2t}$$

and

$$\left|\left\langle\prod_{i=1}^{k}v_{x_i}\right\rangle\left\langle\prod_{j=1}^{k}v_{y_j}\right\rangle\right|\leqslant\tilde{c}_{1,\,k-1}R^k\lambda^{c_2t}. \tag{31}$$

Let $X\nsim Y$ and $X, Y\in t\Omega\cap\mathbf{Z}^{k\nu}$. Let us consider the two possible cases.

1. For all $x_i\neq y_j$ $|x_i-y_j|\geqslant c_2t/2$.
Since $X\nsim Y$, one can find x_{i_0} such that $\rho(x_{i_0},\{x_i\}_{i\neq i_0}\cup\{y_j\}_1^k)\geqslant c_2t/2$. Then by lemma 1

$$\left|\left\langle\prod_{i=1}^{k}v_{x_i}\prod_{j=1}^{k}v_{y_j}\right\rangle\right|=\left|\left\langle\prod_{i=1}^{k}v_{x_i}\prod_{j=1}^{k}v_{y_j}\right\rangle-\langle v_{x_{i_0}}\rangle\left\langle\prod_{i\neq i_0}v_{x_i}\prod_{j=1}^{k}v_{y_j}\right\rangle\right|\leqslant\tilde{c}_{1,\,2k-1}\lambda^{c_2t/2}. \tag{32}$$

2. There are $x_{i_0}\neq y_{j_0}$ such that $|x_{i_0}-y_{j_0}|<c_2t/2$.
Since $\min_{i,\,j}|x_i-x_j|\geqslant c_2t$, $\min_{i,\,j}|y_i-y_j|\geqslant c_2t$, then the distance between $\{x_{i_0}\}\cup\{y_{j_0}\}$ and $\{x_i\}_{i\neq i_0}\cup\{y_j\}_{j\neq j_0}$ is not greater than $c_2t/2$. Since $x_{i_0}\neq y_{j_0}$, then $\langle v_{x_{i_0}}v_{y_{j_0}}\rangle=0$. Then by lemma 1

$$\left|\left\langle\prod_{i=1}^{k}v_{x_i}\prod_{j=1}^{k}v_{y_j}\right\rangle\right|=\left|\left\langle\prod_{i=1}^{k}v_{x_i}\prod_{j=1}^{k}\right\rangle-\langle v_{x_{i_0}}v_{y_{i_0}}\rangle\left\langle\prod_{i\neq i_0}v_{x_i}\prod_{j\neq j_0}v_{y_j}\right\rangle\right|\leqslant\tilde{c}_{2,\,2k-2}\lambda^{c_2t/2}. \tag{33}$$

So, by (31),(32) and (33), we have

$$|(Je_X, Je_Y)|=O(\lambda^{c_2t/2})\qquad\text{as}\quad t\to+\infty. \tag{34}$$

uniformly for $X, Y\in t\Omega$.

Now let $X \sim Y$ and $X, Y \in t\Omega \cap \mathbf{Z}^{kv}$. Recall that $\langle v_{x_1}^2 \rangle = \cdots = \langle v_{x_k}^2 \rangle = 1$. Using $\min_{i \neq j} |x_i - x_j| \geqslant c_2 t$ and lemma 1, we obtain

$$\begin{aligned}\left| \left\langle \prod_{i=1}^{k} v_{x_i} \prod_{j=1}^{k} v_{y_j} \right\rangle - 1 \right| &= \left| \left\langle \prod_{i=1}^{k} v_{x_i}^2 \right\rangle - 1 \right| = \left| \left\langle \prod_{i=1}^{k} v_{x_i}^2 \right\rangle - \prod_{i=1}^{k} \langle v_{x_i}^2 \rangle \right| \\ &\leqslant \left| \left\langle \prod_{i=1}^{k} v_{x_i}^2 \right\rangle - \langle v_{x_1}^2 \rangle \left\langle \prod_{i=2}^{k} v_{x_i}^2 \right\rangle \right| \\ &\quad + |\langle v_{x_1}^2 \rangle| \cdot \left| \left\langle \prod_{i=2}^{k} v_{x_i}^2 \right\rangle - \langle v_{x_2}^2 \rangle \left\langle \prod_{j=3}^{k} v_{x_i}^2 \right\rangle \right| \\ &\quad + \ldots + \prod_{i=1}^{k-2} |\langle v_{x_i}^2 \rangle| \cdot |\langle v_{x_{k-1}}^2 v_{x_k}^2 \rangle - \langle v_{x_{k-1}}^2 \rangle \langle v_{x_k}^2 \rangle| \\ &\leqslant (\tilde{c}_{2,\,2k-2} + \tilde{c}_{2,\,2k-4} + \ldots + \tilde{c}_{2,\,2}) \, \lambda^{c_2 t}.\end{aligned}$$

This computation and (31) imply that for $X \sim Y$

$$(Je_X, Je_Y) = 1 + O(\lambda^{c_2 t}). \tag{35}$$

uniformly in $X, Y \in t\Omega$. Note that $u_{t,X} = u_{t,Y}$ for $X \sim Y$, because $u \in \mathscr{H}^{(k),\, symm}$. The number of $X, Y \in t\Omega \cap \mathbf{Z}^{kv}$ increases as a polynomial in t, hence it follows from (34) and (35) that

$$\begin{aligned}\|Wu\|^2 &= \lim_{t \to +\infty} \left[\sum_{X, Y \in t\Omega \cap \mathbf{Z}^{kv}} (Je_X, Je_Y) \, u_{t,X} \overline{u_{t,Y}} \right] \\ &= k! \lim_{t \to +\infty} \sum_{X \in t\Omega \cap \mathbf{Z}^{kv}} |u_{t,X}|^2 = k! \, \|u\|^2,\end{aligned}$$

which proves (30).

5. ORTHOGONALITY OF $\overline{\text{Ran } W_k}$ AND $\overline{\text{Ran } W_l}$ FOR $k \neq l$

Let $k > l \geqslant 2$ (case $l = 1$ can be verified similarly, case $l = 0$ is trivial). It suffices to prove that for dense subsets $\{u^{(1)}\} \subset \mathscr{H}^{(k),\, symm}$ and $\{u^{(2)}\} \subset \mathscr{H}^{(l),\, symm}$

$$\begin{aligned}(W_k u^{(1)}, W_l u^{(2)}) &= \lim_{t \to +\infty} (W_{k,t} u^{(1)}, W_{l,t} u^{(2)}) \\ &= \lim_{t \to +\infty} (J_k \exp(itL^{(k)}) \, u^{(1)}, J_l \exp(itL^{(l)}) \, u^{(2)}) = 0.\end{aligned}$$

As before, choose $\hat{u}^{(1)} \in C^\infty(T^{kv})$, $\hat{u}^{(2)} \in C^\infty(T^{lv})$ such that $\nabla_{p_i} M_k(P^{(k)}) \neq \nabla_{p_j} M_k(P^{(k)})$ for $P^{(k)} \in supp\, \hat{u}^{(1)}$, $\nabla_{p_i} M_l(P^{(l)}) \neq \nabla_{p_j} M_l(P^{(l)})$ for $P^{(l)} \in supp\, \hat{u}^{(2)}$ and Ω_1, Ω_2 open bounded sets, corresponding to $\hat{u}^{(1)}$, $\hat{u}^{(2)}$. By lemma 3

$$(J_k \exp(itL^{(k)})\, u^{(1)}, J_l \exp(itL^{(l)})\, u^{(2)})$$

$$= \sum_{\substack{X \in t\Omega_1 \cap \mathbf{Z}^{kv} \\ Y \in t\Omega_2 \cap \mathbf{Z}^{lv}}} u^{(1)}_{t,X} \overline{u^{(2)}_{t,Y}} (J_k e_X, J_l e_Y) + o(1), \qquad t \to +\infty \tag{36}$$

For sufficiently large t if $X \in t\Omega_1 \cap \mathbf{Z}^{kv}$, $Y \in t\Omega_2 \cap \mathbf{Z}^{lv}$ then $X \notin S_k$, $Y \notin S_l$ and

$$(J_k e_X, J_l e_Y) = \left\langle \prod_{i=1}^{k} v_{x_i} \prod_{j=1}^{l} v_{y_j} \right\rangle - \left\langle \prod_{i=1}^{k} v_{x_i} \right\rangle \left\langle \prod_{j=1}^{l} v_{y_j} \right\rangle. \tag{37}$$

As before, the second summand is $O(\lambda^{ct})$ for some $c > 0$ and $\lambda < 1$. Let us bound the first one. Let $c_2^{(1)} = \inf_{i \neq j,\, X \in \Omega_1} |x_i - x_j| > 0$. If $X \in t\Omega \cap \mathbf{Z}^{kv}$ and $Y \in \mathbf{Z}^{lv}$, then for some i_0 $\rho(x_{i_0}, \{x_i\}_{i \neq i_0} \cup \{y_j\}) \geqslant c_2^{(1)}/2$, since $l < k$. It follows then by lemma 1 that the first summand in (37) is $O(\lambda^{c_2^{(1)} t/2})$. Since the number of $X \in t\Omega_1$ and $Y \in t\Omega_2$ increases as a polynomial in t, the r.h.s. in (36) tends to 0.

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